Categorical Propositions and the Traditional versus the Modern Square of Opposition in Terms of Sets and Set Operations

Here we examine the standard categorical propositions and express them using the language and notation of sets. Key concepts will be set intersection, subsets, set compliment, set difference and the empty set. Make sure you have reviewed these concepts and are familiar with them before continuing. You should go through the exercises at the end of each section to solidify your knowledge of the material.

1.0 Categorical Propositions in Standard Form

Definition: A **categorical proposition** in *standard form* is a proposition in one of the following forms:

All S are P
No S are P
Some S are P
Some S are not P

The words, "all", "no", and "some", are called the standard **quantifiers**. The variable S is called the **subject term**, and P the **predicate term**. In the above examples the verb is always the word "are" but sometimes "is" can be used as appropriate. The verb is called the **copula**, since it couples together the subject and predicate terms. We invite the student to abstract beyond the standard numerical quantifiers listed above to include words like, "always", "every one", "none", "never", "sometimes", "at least one", etc. In general, what we discover below will hold also for any similar substitution whose scope is everything in the class (*universal*) or whose scope is at least one (*particular*). Many texts prefer the use of the phrase, "at least one" instead of "some". You are free to use whichever or both in what follows.

The variable S stands for the subject term, and the variable P the predicate term**.** The subject term S should be thought of as the set of things which have the property S. The predicate term P should be thought of as the set of all things which have property P. In other words P and S are the sets of all things which have the stated properties P and S respectively. We will not place any restrictions on these properties for the sake of simplicity. However, an unrestricted approach to "allowable properties" eventually results in some notorious problems, but for this elementary approach we will ignore those problems.[1]

To say, for example, "All S are P" means that everything included in the set S has property P. For our treatment we will always assume that P is a non-empty set. This is not an unwarranted assumption, since if we wish to show that nothing exhibits a

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[1] See, for example Russell's Paradox http://en.wikipedia.org/wiki/Russell%27s_paradox

given property P (say the property of being colorful and colorless at the same time), we can express this by stating "No S are P" for any possible choice of S, which turns out to be equivalent to saying that the set P is empty (see exercise 6.3). In general, it is probably better to think of P as not a set, but a **class**, which has a different meaning than a set, but since our aim is not to delve deeply into set theory, axiomatic or otherwise, we leave this consideration aside at this level where it will not pose any problem and continue our exploration.

For convenience, we introduce variables which stand for each of the four mentioned propositions. The choice of the following variables has historical roots and are standard in the treatment of categorical propositions. Take note of these, as they will be used throughout the remainder of the text.

All S are P	A
No S are P	E
Some S are P	I
Some S are not P	O

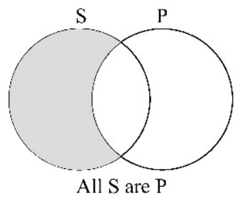
Exercises 1.0

1. Let the subject term S be "tests" and let the predicate term P be "things difficult to pass". Write out the corresponding A, E, I, and O propositions.
2. Repeat exercise 1 using "always" instead of "all", "never" instead of "no" and "sometimes" instead of "some".
3. Repeat exercise 1 using "are not" instead of "are" for the copula for the A and E propositions. Make a statement about the resulting change in meaning for both statements.
4. Suppose "Some S are P" is a true statement, does this mean "Some S are not P" must also be a true statement? Write out as many diverse examples as possible to help guide your reasoning. State clearly the reasoning behind your answer.

1.1 Categorical Propositions expressed in terms of Sets

Our goal is to express categorical propositions in terms of sets, and to use theorems about sets and set operations to discover valid inferences which can be obtained from these categorical propositions.
The set of valid inferences between the A, E, I and O propositions will depend on how we define the terms, "All S are P" and "No S are P". These definitions will be called the **Boolean** and the **Aristotelian** definitions respectfully.
To begin we will naively set aside these two different definitions, which we have only mentioned, and we will assume the subject term S is the set of all things which are S or have the property S, whichever makes most sense for the specific example. Hence in, "All men are mortal", the subject term is "men" hence S will be the set of

things which have the property of being men. The same will be true of the predicate term P, so in this case P will be the set of all things which have the property of being mortal.



1.1.1 All S are P

This categorical proposition states that everything that is an S is also a P. This can be expressed in the language of sets by saying that S is a subset of P, which is expressed symbolically as $S \subseteq P$. Hence we have:

> The categorical proposition, "All S are P" is expressed in set notation as,
>
> $$S \subseteq P$$

However, we can express the same idea using the notion of intersection of sets, which will be more useful when we do operations on sets in order to discover valid inferences. In this case, since everything in S is also in P, then the intersection of S with P is just S. This gives the following useful and equivalent notation for, ""All S are P". Updating our definition we now have:

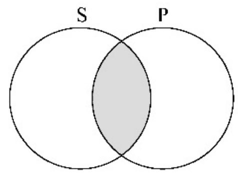
> The categorical proposition, "All S are P" is expressed in set notation as,
>
> $$S \subseteq P \text{ or } S \cap P = S$$

Exercises 1.1.1

1. Let S = {1,2,3,5,7} and let P = {even numbers}.
a) Is $S \subseteq P$? Why or why not?
b) Is the statement, "All S are P" true?

2. Let S = {dogs}, and let P = {animals with a tail}.
a) What is $S \cap P$?
b) Is the statement, "All S are P" true?

3. Let S be the empty set, and let P be the empty set.
a). Is $S \subseteq P$? Why or why not?
b) Is the statement, "All S are P" true?



No S are P

1.1.2 No S are P

This categorical proposition states that nothing that is an S is also a P. We can express this idea using the notion of intersection of sets. In this case, since nothing in S is also in P, then the intersection of S with P is the empty set, $\emptyset$. This gives the following useful and equivalent notation for, ""No S are P".

> The categorical proposition, "No S is P" is expressed in set notation as,
>
> $$S \cap P = \emptyset$$

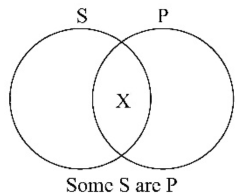
However, in this case, the categorical proposition, "No S are P" can **not** be expressed as, $S \subsetneq P$ (S is not a subset of P), exercise 1 explores why this is the case.

Exercises 1.1.2

1. Let S = {1,2,3,5,7} and let P = {even numbers}.
a) Is $S \subsetneq P$? Why or why not?
b) What is $S \cap P$?
c) Is the statement, "No S are P" true?

2. Let S = {mammals with tails}, and let P = {ants}.
a) What is $S \cap P$?
b) Is the statement, "No S are P" true?

3. Let S be the empty set, and let P be the empty set.
a). What is $S \cap P$?
b) Is it also true that $S \cap P = S$?
c) Is the statement, "No S are P" true? Is the statement, "All S are P" true?



1.1.3 Some S are P

This categorical proposition states that *at least one member* of the set S is also in P. This means that at least one element of S *exists*, and that it has property P. This can be expressed in the language of sets by saying that the intersection of S and P is not empty. Hence we have:

> The categorical proposition, "Some S are P" is expressed symbolically in set notation as,
>
> $$S \cap P \neq \emptyset$$

Notice how nice this notation is. We do not have to stipulate that the set S is non-empty, for if it were, then $S \cap P = \emptyset$, but our statement says that their intersection is not empty whenever "Some S are P" is a true statement!

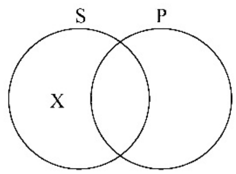
Exercises 1.1.3

1. Let S = {1,2,3,5,7} and let P = {even numbers}.
a) What is $S \cap P$?
b) Is the statement, "Some S are P" true?

2. Let S = {mammals with tails}, and let P = {ants}.
a) What is $S \cap P$?
b) Is the statement, "Some S are P" true?

3. Let S be the empty set, and let P be the empty set.
a). What is $S \cap P$?
b) Is the statement, "Some S are P" true?

1.1.4 Some S are not P



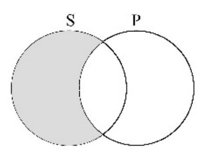
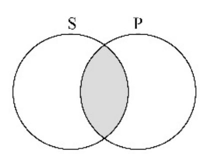
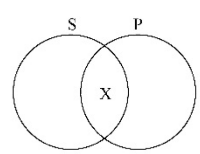
Some S are not P

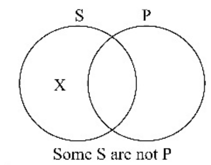
This categorical proposition states that at least one S is not P. This means that at least one element of S exists, and that it is not an element of the set P. This can be expressed in the language of sets by saying that the intersection of S and *the compliment of* P, which we will denote as P^C, is not empty. Hence we have:

> The categorical proposition, "Some S are not P" is expressed symbolically in set notation as,
>
> $$S \cap P^C \neq \emptyset$$

Knowing the definition of the compliment of P will be helpful. Recall that U is our assumed Universe (where all our sets reside), then the compliment of P is simply everything in the set U which is not in P, or, $P^C = U - P$ (which is sometimes expressed as $P^C = U \backslash P$).

Summary of Categorical Propositions in Terms of Sets

Categorical Proposition	Equivalent statement in sets	Common graphic Representation	Traditional Shorthand Letter
All S are P	$S \subseteq P$ or $S \cap P = S$	 All S are P	A
No S are P	$S \cap P = \emptyset$	 No S are P	E
Some S are P	$S \cap P \neq \emptyset$	 Some S are P	I

Some S are not P	$S \cap P^C \neq \emptyset$		O

You should become **very familiar** with these equivalent ways of expressing categorical propositions. We will often move from a statement like "No S are P" to the equivalent statement $S \cap P = \emptyset$ *without reminding you that both statements say the same thing*!

Exercises 1.1.4
In this set of exercises, the Universal set U will be all positive whole numbers less than 20, or in list form we have
U = {1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19}.

1. Let S = {1,2,3,5,7} and let P = {**odd** numbers less than 10}.
a) Is $S \neq \emptyset$?
b) What is P^C?
c) What is $S \cap P^C$?
d) Is the statement, "Some S are not P" true?

2. Let S = {1,2,3,5,7} and let P = U.
a) What is $P^C = U - P$?
b) Is P^C the same as $U - U$ in this case?
c) What is $S \cap P^C$?
d) Is the statement, "Some S are not P" true?

2.0 Immediate Inferences and the Boolean and Aristotelian Viewpoints

Let us consider how *immediate inferences* differ according to the Boolean and Aristotelian viewpoints.

Definition: An **immediate inference** is a valid inference that can be made from a single statement.

An example of an immediate inference is the inference that "John is no longer a teenager" from the statement that "John was born in 1994".

Now consider the following universal proposition:

All people from Mars are amphibians

Supposing this statement is true, is the following an immediate inference?

Some people from Mars are amphibians.

Let's answer this question by looking at our definitions of these terms using sets. Recall that "All people from Mars are amphibians" is equivalent to $M \cap A = M$ and "Some people from Mars are amphibians" is equivalent to $M \cap A \neq \emptyset$. Hence our statement can be asked as to whether the truth of

$$M \cap A = M$$

guarantees the truth of

$$M \cap A \neq \emptyset$$

But suppose there are no Martians (the set of Martians is empty, e.g. $M = \emptyset$), then the first statement is still true, as $M \cap A = M$ is just $\emptyset \cap A = \emptyset$ which follows from theorem 8 found in *Sets. Basic Operations, Theorems and Examples*. But since M is empty, this also implies that $M \cap A = \emptyset$ which has the opposite truth value of $M \cap A \neq \emptyset$. Hence in this case, the inference from $M \cap A = M$ to $M \cap A \neq \emptyset$ is invalid.

Now let's consider another example. Let's keep the same question, but change the subject term M to "salamanders". Following the same line of reasoning above we have the truth of $M \cap A = M$ implying the truth of $M \cap A \neq \emptyset$, because the set of salamanders is not empty!

When we introduced the expression of "All S are P" and "No S are P" above, we did not worry about whether the set S could be empty or not. Let us now state two different definitions of the A and E propositions, the **Aristotelian** and **Boolean** definitions.

Definition (Aristotelian): Let S be any *non-empty set*, then the statement, "**All S are P**" is equivalent to $S \subseteq P$ or $S \cap P = S$.

Definition (Aristotelian): Let S be any *non-empty set*, then the statement, "**No S are P**" is equivalent to $S \cap P = \emptyset$.

Notice that the definitions require S to be a *non-empty set*. This is what is meant when we say the Aristotelian interpretation has ***existential import*** – it is just the assumption that the subject class S be non-empty.[2]

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[2] Some texts state rather oddly that the A categorical proposition implies the existence of its subject class, *if the subject class exists*. But when the subject class is known not to exist then no such implication is made. The same is said of the E categorical proposition. Such a circular definition often leaves the student scratching their heads. See, for example, Hurley, Patrick. A Concise Introduction

Now we can re-examine the following inference (many times referred to as **subalternation**, see below) using the Aristotelian definition.

All S are P

We can now infer the truth of the statement

Some S are P

Now looking at our definitions of these terms using sets, we have the truth of

$$S \cap P = S$$

guaranteeing the truth of

$$S \cap P \neq \emptyset$$

Notice that in this case, the set S can not be empty, thus the truth of $S \cap P = S$ implies the truth of $S \cap P \neq \emptyset$, *because* S is not allowed to be empty.

When the validity of any inference made by means of categorical propositions rests on which definition is used at some point in the inference, then we say the inference commits the **existential fallacy**. Note that this does not mean that the inference is wrong, but rather one needs to check whether the subject class exists to make sure the inference is indeed valid.

We now turn to the Boolean interpretation, which is the **default definition** and the one implied at the beginning of this treatment. Since it is customary in discussing sets that if no conditions are specified in advance then the sets under discussion could be empty as well as non-empty, this leads to the Boolean definitions.

Definition (Boolean): Let S be any set (empty or nonempty), then the statement, "**All S are P**" is equivalent to $S \cap P = S$ or $S \subseteq P$.

Definition (Boolean): Let S be any set (empty or nonempty), then the statement, "**No S are P**" is equivalent to $S \cap P = \emptyset$.

Notice that in the second case, S can be any set, including sets which are empty, hence under the Boolean interpretation, we can not assume anything about the existence of elements of S, since S can be the empty set. By custom the expression,

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to Logic. 12th edition. Cengage Learning, Stamford CT. 2015, page 209. Things become more clear when we examine such statements in terms of sets.

"let S be any set" is omitted unless such omission might cause confusion. I have inserted it here to emphasize the difference between the Boolean and the Aristotelian definitions.
Also note that the Boolean and Aristotelian definitions only apply to the A and E propositions. As the truth of both the I and O propositions already assumes that the set S is non-empty.

This difference in Boolean and Aristotelian interpretations is often expressed in terms of what is called *existential import*. However, the important question is whether the subject class is empty or non-empty, hence we will leave the treatment of existential import to those textbooks which approach this subject by some other means than sets.

Exercises 2.0 (assume the Boolean definitions)

1. Let S = {2,14,26,198} and let P = {even numbers}
a) What is $S \cap P$?
b) Is $S \cap P$ empty?
c) Does the truth of $S \cap P = S$ guarantee the truth of $S \cap P \neq \emptyset$?
2. Let S = {things which are both perfectly square and perfectly round at the same time}, and let P = {Set of all geometric figures}
a) What is $S \cap P$?
b) Is $S \cap P$ empty?
c) Does the truth of $S \cap P = S$ guarantee the truth of $S \cap P \neq \emptyset$??

3.0 Relations Between Statements

In studying categorical propositions, we often encounter the following important relationships between the truth of two or more propositions. We define each one individually and then discuss two which are of particular importance when discussing categorical propositions.

Definition: Two statements are **contradictory** if exactly one of the statements must be true. When this happens we say the statements are **contradictions**.

Notice that if P and Q are contradictory statements, then **exactly one** must be true (hence) the other must be false. In general if P is any statement, no matter how complex, *then negating the entire* statement produces the contradiction to P. We can do this by just writing, "It is not the case that" in front of our statement P, but many times we want other equivalent and more informative ways of stating the contradiction to P.

Example: Let P be "Today is Monday" and let Q be "Today is not Monday", then P and Q are contradictory.

Definition: Two or more statements are **contraries** if the truth of one statement implies that all the other statements are false, and it is possible for all statements to be false.

Observe that when a set of statements are contraries, if one of them turns out to be true all the others must be false. However it is possible for all the statements to be false.

Example: Let P be "Today is Monday", let Q be "Today is Thursday" and let S be "Today is Sunday" then P, Q and S are contrary statements. If one is true all the others must be false, however if today is Tuesday, all are false.

Definition: Two or more statements are **subcontraries** if the falsity of one statement means all the other statements must be true. However it is possible for all statements to be true.

Observe that subcontraries can be obtained from a set of contraries by negating each statement.

Example: Let P be "Today is not Monday", let Q be "Today is not Thursday" and let S be "Today is not Sunday" then P, Q and S are subcontrary statements. If one is false, (for example today is Sunday) then all the rest must be true. However if today is Tuesday, then P, Q and S are all true.

Definition: Given two statements P and Q, then P is said to **imply** Q if the truth of P guarantees the truth of Q, or stated other way, it is impossible for P to be true and Q to be false.

Example: Let P be Today is Monday, and let Q be Today is not Friday, then P implies Q.

We should observe that **implication** is just another common word for **entailment** and also just another word that refers to the logical operation found in the conditional statement "If P then Q". We also draw attention to the fact that the following two statements are logically equivalent (always have the same truth value):

"*If P then Q*" is logically equivalent to "*If Not Q then Not P*"

Many Logic textbooks combine both of these facts into one relationship called "**subalternation**" . By introducing an arrow of implication, "$\Longrightarrow$" whose meaning is identical with the " $\supset$ " symbol used in many textbooks, we can observe the following :

Definition: Given a true implication, $P \implies Q$, then we say the statements P and Q are **subalternates**, since if P is true, then Q must be true, and if Q is false, P must also be false.

Sometimes the above relationship is explained by saying that truth follows the direction of the arrow, and falsity follows in the reverse direction.
Since we already have the concept of "implication" we will not use the term "subalternation" in this treatment. However, since it is used in many textbooks when discussing categorical propositions, I have included it here for reference.

Exercises 3.0
Determine the type of relation that exists between the following sets of propositions P and Q. If the relationship is that of implication write the arrow of implication. In other words, if P implies Q, then write $P \implies Q$, if Q implies P, then write $Q \implies P$. For those familiar with the "" sign for implication rather than the " $\implies$ ", feel free to use it.

1. Let P be "Today is not Friday" and let Q be "Today is Monday"
2. Let P be "Jane is not 31" and let Q be "Jane is 31"
3. Let P be "Today is Monday" let Q be "Today is Thursday"
4. Let P be "Jane is not 33" and let Q be "Jane is not 23"
5. Let Q be "Today is not Monday" and let P be "Today is Friday"

4.0 Contradictions between the A and E Categorical Propositions and the Modern Square of Opposition.

Let us pose a simple but revealing question. Consider the following two statements:

P = "Some students passed the class"

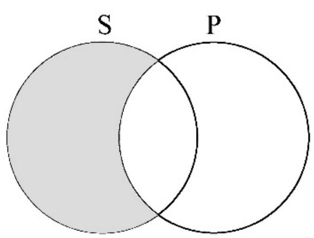
Q = "Some students did not pass the class"

Seemingly the truth of one tells us something about the truth of the other, but what? At first sight they seem to be contradictions because the main verb is negated, an action which many times produces a contradiction. Let's test that suggestion by considering the following possibility:

Suppose there are 100 students in class, 51 passed the class, and 49 failed to pass. It is correct to claim that "Some students passed the class" – but to claim that "Some students did not pass the class" must be false (which must be the case if the two sentences are contradictions), is incorrect. Hence in this case the two statements are not contradictions, as they fail to satisfy the definition. What then is the contradiction to "Some students passed the class" and, in general what is the contradiction to A and E categorical propositions?

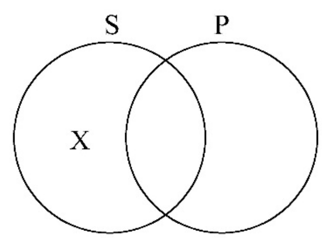
One way to obtain the answer comes from negating our Universal A and E propositions as expressed in sets – which is the theme of this paper, so that is how we will proceed.

For our A proposition, we discovered “All S are P” is equivalent to $S \cap P = S$, hence its contradiction is equivalent to $S \cap P \neq S$ (in words, “the intersection of S with P equals S” has for its contradiction, “the intersection of S with P does *not* equal S”). Since the intersection of S with P contains all the elements in common with S, and the assertion is that the intersection does not equal S, then there is at least one element in S which is not also in P, which means that the intersection of S with P^C is non-empty, but this is equivalent to the assertion that “Some S are not P”. In pictures, we have:



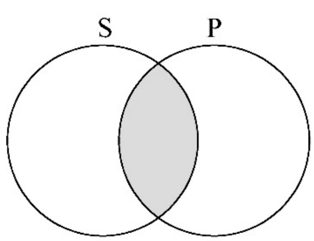
All S are P

is contradicted by



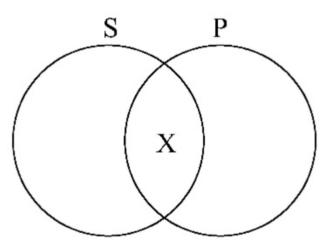
Some S are not P

For our E proposition, we discovered “No S are P” is equivalent to $S \cap P = \emptyset$, hence its contradiction is equivalent to $S \cap P \neq \emptyset$, (in words, “the intersection of S with P is empty” has for its contradiction, “the intersection of S with P is not empty”). This means there is some element of S that is contained in the intersection of S with P, or, “Some S are P”. In pictures we have:



No S are P

is contradicted by

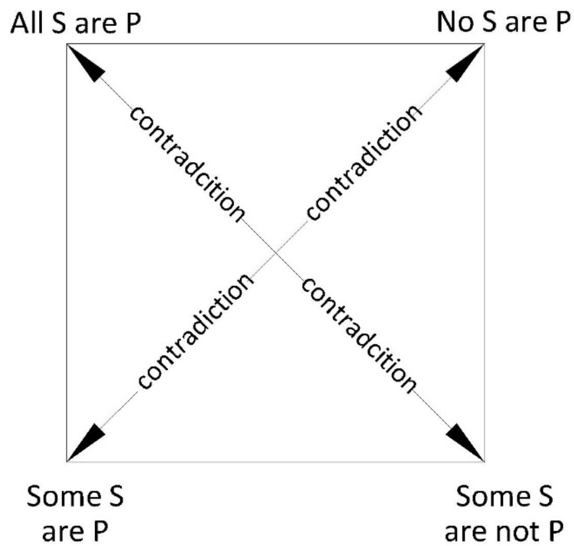


Some S are P

Clearly these relationships hold when we use the Aristotelian and Boolean definitions for the A and E propositions (just do a quick check, recall that “Some S are P” and “Some S are not P” both require the set S to be non-empty, so the only thing we need to check is whether “All S are P” and “No S are P” make sense when S is empty).

These relationships are often expressed in what is often called the Boolen or Modern Square of opposition, as shown below:

The Modern/Boolean Square of Opposition



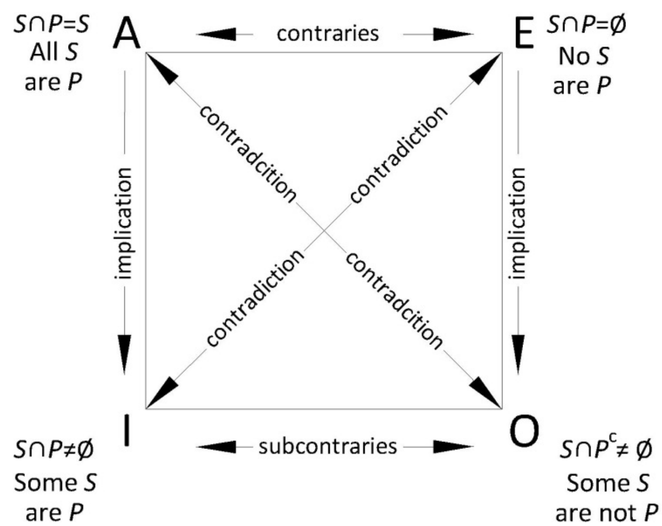
Exercises 4.0
Use the square of opposition to write to contradiction to the following categorical propositions.

1. All student passed the class.
2. Some days were not accounted for by the journal entries.
3. No person was found to hold an illegal passport.
4. Some cars were recalled by the manufacturer.
5. Are the following two sentences contradictions? Give reasons why or why not.
   Sometimes winters are colder than normal.
   Winters are never colder than normal.
6. (harder) Are the following two sentences contradictions? Give reasons why or why not.
   All students passed the course.
   Many students did not pass the course.

5.0 The Traditional Square of Opposition

Suppose we insist upon using the Aristotelian definition for the universal categorical propositions A and E, which means that the subject classes are not empty. The result is that several more immediate inferences appear on the square of opposition. We present these inferences below, and prove some of them and leave the proofs for the rest as exercises.

The Traditional/Aristotelian Square of Opposition



First we will give informal reasons why these relationships are true, and then leave the formal proofs as an advanced reading for those who wish to go further and prove things formally.

To show that these relationships exists informally, we will just define sets S and P (which are not in any way special), and show they work for these specific sets. By doing so, and noting that the sets we use are not atypical or special, we claim that this gives good reasons to suspect that these relationships will hold for any sets such that the subject class S is non-empty. In sections 5.1 and 5.2 we provide formal proofs that this is indeed the case.

First the sets! Let $S = \{2, 4, 6, 8, 10\}$ and let $P =$ $\{\text{even numbers less than 20 but greater than zero}\}$.

First observe that all S are indeed P since every element in S is also included in P. This in turn implies that "Some S are P". Notice how odd this sounds, since we are accustomed to thinking that if some S are P, then some S are not P, but with "some S are P" expressed as $S \cap P \neq \emptyset$, its truth becomes clear.

Clearly $S \cap P \neq \emptyset$ and $S \cap P = \emptyset$ are contradictions, as one asserts that the intersection of S with P is not empty, whereas the other says it is.

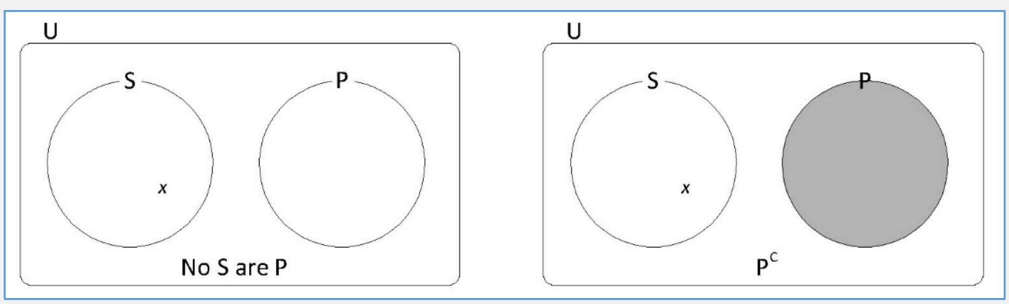
That both $S \cap P = S$ and $S \cap P = \emptyset$ can not both be true (under the assumption that S is not empty) is clear, irrespective of any choice of S. To show both can be false, we need another choice for S (since the current one makes "all S are P" true, and we need a case where it is false). For this we try $S = \{-2, 0, 6, 8, 10\}$ and leave P unchanged. In this case, "All S are P" is false, and so is the statement "No S are P", since $S \cap P = \{6, 8, 10\}$ which is clearly not empty. This shows that the A and I propositions are indeed contraries in this case.

Exercises 5.0 (assume the Aristotelian definitions)
1) Use a similar argument to the one given above to show "No S are P" implies that "Some S are not P". (hint: try $S = \{1, 3, 7, 9, 11\}$ and $P = \{\text{even numbers less than } 20\}$, with the Universal Set $U = \{\text{whole numbers greater than zero but less than } 20\}$ (remember we need a Universal Set when we deal with the compliment of a set).

2) Explain how the following "visual" proof is another way one can argue for exercise 1.

Theorem: If the set S is non-empty, then the statement "No S are P" implies the statement "Some S are not P"

Visual Proof:



Hint: Recall *x* just represents an arbitrary element of S, which we know exists since S is non-empty. The picture on the right illustrates $S \cap P^c$, where shaded areas are considered empty.

3) Show that "Some S are P" and "Some S are not P" are subcontraries by choosing a set S and another set P such that both can not be false, but both can be true. (hint, one of the above choices for S and another choice of P above will work).

4) Show that the A and O propositions are contradictions. (Hint, clearly the contradiction to $S \cap P^c \neq \emptyset$ is $S \cap P^c = \emptyset$, so all you need to show is that $S \cap P^c = \emptyset$ is equivalent to $S \cap P = S$. Try:
$S = \{2, 4, 6, 8, 10\}$, $P = \{\text{even numbers less than 20 but greater than zero}\}$ and $U = \{\text{whole numbers greater than zero but less than 20}\}$

6.0 Advanced Reading

In this section we will prove some of the above observations by using the basic theorems about sets as given in the handout, Sets. Basic Definition, Theorems, Operations and Examples. These theorems are listed again in the Appendix. Our first Theorem will be to prove formally that the A and E propositions are not necessarily contraries under the Boolean definition (which allows the subject class to be empty).

Theorem 1: The A and E propositions are not necessarily contraries under the Boolean definition.

We want to show that at least one criterion for two propositions to be contraries given by the definition does not hold. In this case, we will show that both $S \cap P = S$ and $S \cap P = \emptyset$ can be true. Since our theorem deals only with the Boolean interpretation, the first line in the proof re-iterates this.

Proof: Suppose we are using the Boolean definition and that the subject class S is empty. Consider the statement $S \cap P$, we want to show $S \cap P = \emptyset$. Since $S = \emptyset$ we have,

$$
\begin{aligned}
S \cap P &= \emptyset \cap P && \text{since we assumed } S = \emptyset \\
&= P \cap \emptyset && \text{by the commutative laws} \\
&= \emptyset && \text{by theorem 8} \\
&= S && \text{since } S = \emptyset
\end{aligned}
$$

Hence $S \cap P = \emptyset$ and $S \cap P = S$ which means both the A and E propositions can be true, which would be impossible if they were contraries.

Notice that the above theorem used the fact that the Boolean interpretation allows for S to be empty, and also uses one of the definitional properties of two statements to be contraries to show that under the Boolean definition, the A and E propositions need not be contraries (in fact the only time they are not is when the subject class S is empty, and this is precisely what is allowed under the Boolean interpretation).

Next we want to prove formally that the A and O propositions are indeed contradictions under both interpretations. We did this informally in exercise 5.0.4, where we noted that we needed to show that $S \cap P^c = \emptyset$ is equivalent to $S \cap P = S$. We now prove this formally.

Theorem 2: $S \cap P^c = \emptyset$ is equivalent to $S \cap P = S$

We do this by applying the same set operations to each side of the equality and using basic theorems found in Sets. Basic Definition, Theorems, Operations and Examples until we show that $S \cap P^c = \emptyset$ is equivalent to $S \cap P = S$

Proof:

$S \cap P^c = \emptyset$	This is our assumption
$(S \cap P^c)^c = \emptyset^c$	we take the compliment of both sides
$S^c \cup P = U$	by DeMorgan's and theorem 3
$(S^c \cup P) \cap S = U \cap S$	we intersect both sides with the set S
$(S^c \cap S) \cup (S \cap P) = S$	by distribution and theorem 7
$\emptyset \cup (S \cap P) = S$	since $S^c \cap S = \emptyset$ by theorem 9
$S \cap P = S$	by theorem 12

Hence $S \cap P = \emptyset$ is equivalent to $S \cap P = S$ which means the A and O propositions are indeed contradictions. Also notice that our proof never required that S be non-empty, this shows they are contradictions under both the Aristotelean and Boolean definitions.

With a definition of *implication* in terms of sets and set operations, we could proceed further, but that takes us too far afield of this elementary approach. We leave this treatment with three advanced exercises.

Exercises 6.0

1) Follow the example of Theorem 1 to prove that "Some S are P" and "Some S are not P" need not be subcontraries under the Boolean definition by showing that both can be false.

2) For this exercise we will drop the assumption that the predicate class P be non-empty and assume that $P = \emptyset$. Now, show that under the Aristotelian definition, the A and E propositions need not be contraries by showing it is not possible for both statements to be false. Explain why, under the Aristotelian definition and the assumption that $P = \emptyset$, that one of the A and E propositions must be false, and the other one must be true which makes the statements contradictions rather than contraries.

3) Suppose the statement $S \cap P = \emptyset$ is *true* for ***any choice*** of set S . Give reasons why this is equivalent to the assumption that $P = \emptyset$ and that in this case, “No S are P” must be true.

Kent Slinker
San Antonio College/Arizona Western College
kslinker@alamo.edu
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Appendix

Some basic Theorems about Sets

1. $(P^C)^C = P$ The compliment of a set compliment is the original set.
2. $U^C = \emptyset$ The compliment of the Universal set is the empty set.
3. $\emptyset^C = U$ The compliment of the empty set is the Universal set.
4. $P - \emptyset = P$ The empty set subtracted from any set is just that set.
5. $P - P = \emptyset$ Any set subtracted from itself is the empty set.
6. $P - S = P \cap S^C$ S subtracted from P is the intersection of P with the compliment of S.
7. $P \cap U = P$ The intersection of P and the universal set is P.
8. $P \cap \emptyset = \emptyset$ The intersection of P and the empty set is the empty set.
9. $P \cap P^c = \emptyset$ The intersection of P and the compliment of P is the empty set.
10. $P \cap P = P$ The intersection of P with itself is P
11. $P \cup U = U$ The union of P and the universal set is P.
12. $P \cup \emptyset = P$ The union of P and the empty set is P.
13. $P \cup P^c = U$ The union of P and the compliment of P is the universal set U.
14. $P \cup P = P$ The union of P with itself is P.

Commutative Laws

15. $P \cup S = S \cup P$ The operation of taking unions commutes.
16. $P \cap S = S \cap P$ The operation of taking intersections commutes.

Associative Laws

17. $P \cup (S \cup R) = (P \cup S) \cup R$ The operation of taking unions is associative.
18. $P \cap (S \cap R) = (P \cap S) \cap R$ The operation of taking intersections is associative.

Distributive Laws

19. $P \cup (S \cap R) = (P \cup S) \cap (P \cup R)$ The set P distributes across the set $S \cap R$
20. $P \cap (S \cup R) = (P \cap S) \cup (P \cap R)$ The set P distributes across the set $S \cup R$

DeMorgan's Laws for Sets

21. $(P \cup S)^C = P^C \cap S^C$
22. $(P \cap S)^C = P^C \cup S^C$
23. $P - (S \cup R) = (P - S) \cup (P - R)$
24. $P - (S \cap R) = (P - S) \cup (P - R)$

Answers to Exercises

Answers Exercises 1.0

1. The A proposition: "*All tests are things difficult to pass*"
The E proposition: "*No tests are things difficult to pass*"
The I proposition: "*Some tests are things difficult to pass*", or equivalently, "*At least one test is a thing which is difficult to pass*"
The O proposition: "*Some tests are things not difficult to pass*", or equivalently, "*At least one test is a thing which is not difficult to pass*"

2. The A proposition: "*Always tests are things difficult to pass*"
The E proposition: "*Never are tests things difficult to pass*"
The I proposition: "*Sometimes tests are things difficult to pass"*
The O proposition: "*Sometimes tests are things which are never difficult to pass"*

3. "*All tests **are not** things difficult to pass*" (means the same as "*No tests are things difficult to pass*")
"*No tests **are not** things difficult to pass*" (means the same as "*All tests are things difficult to pass*")

So by changing the copula from "are" to "are not" the A proposition becomes the same as the E proposition and the E proposition becomes the same as the A proposition.

4. Consider the statement, "All Swans are white", and suppose this is actually true. If we accept that the truth of this statement means, "Some swans are white" is also true, then the truth of "Some swans are white" does not ensure that "Some swans are not white".
In other words, if we accept that the truth of "All S are P" entails the truth of "Some S are P", then in this case it will be false that "Some S are not P" is also true.

This is covered in more detail in subsequent sections of this reading.

Answers Exercises 1.1.1

1.
a) A set S is a subset of P if and only if every element of S is also an element of P. Clearly S contains odds numbers, whereas P only contains even numbers.
b) No, since S contains numbers which are not even.

2
a) $S \cap P = \{\text{dogs}\}$ since the set of dogs is a subset of animals with tails.

b) Yes, since every dog is also an animal with a tail.

3.
a) Yes, S is a subset of P, since the empty set is a subset of every set (See "*Sets. Basic Definitions, Operations, Theorems and Examples"*)
b) Yes since since the statement, $S \subseteq P$ becomes $\emptyset \subseteq P$ since $S = \emptyset$ and by Theorem 8 and the Commutative laws we have: $\emptyset \subseteq P = \emptyset = S$, hence "All S are P" is true.

Answers Exercises 1.1.2

1.
a) S is not a subset of P, since there are elements in S which are not in P.
b) $S \cap P = \{2\}$
c) The statement "No S are P" is false, since $S \cap P = \{2\} \neq \emptyset$

2.
a) $S \cap P = \emptyset$
b) Yes, since no mammal is an ant.

3.
a) $S \cap P = \emptyset$ by Theorem 8
b) Yes, since it is given that $S = \emptyset$
c) "No S are P" is true, since $S \cap P = \emptyset$, but it is also the case that "All S are P" is true also , since $S \cap P = S = \emptyset$.

Answers Exercises 1.1.3

1.
a) $S \cap P = \{2\}$
b) Yes, since at least one member of the set S is also in the set P.

2.
a) The empty set.
b) No, since the intersection of the empty set with any set is just the empty set, so $S \cap P \neq \emptyset$ is not a true statement.

3.
a) In this case, as before, $S \cap P = \emptyset$
b) If "Some S are P" were true, then we would have $S \cap P \neq \emptyset$, but since S is the empty set, this is not a true statement, so it is not true that "Some S are P".

Answers Exercises 1.1.4

1.
a) Since S contains the numbers 1,2,3,5, and 7 then S is not empty.
b) Observe that $P = \{1,3,5,7,9\}$ so given our stated universal set U, we have $P^c = \{2,4,6,8,10,11,12,13,14,15,16,17,18,19\}$
c) Since S = {1,2,3,5,7}, then $S \cap P^C = \{2\}$
d) Yes, since $S \cap P^C = \{2\}$, and {2} is not the empty set.

2.
a) In this case since we are given that $P = U$, then by Theorem 2 we have $P^c = U^c = \emptyset$
b) Yes since by Theorem 5, we have $U - U = \emptyset$
c) Since $P^c = \emptyset$ from part a) above, by Theorem 8 we have $S \cap P^C = S \cap \emptyset = \emptyset$
d) No the statement, "Some S are not P" is not true, since $S \cap P^C = \emptyset$

Answers Exercises 2.0

1.
a) $S \cap P = \{2,14,26,198\} = S$
b) No, $S \cap P$ is not empty since S is not empty.
c) Since S is not empty, then yes, $S \cap P = S$ implies that $S \cap P \neq \emptyset$.

2.
a) Observe that since no object has the property of being "both perfectly square and perfectly round at the same time" the set S is empty. Hence $S \cap P = \emptyset \cap P = \emptyset$
b) Yes
c) No, (it actually implies *the falsity* of $S \cap P \neq \emptyset$)

Answers Exercises 3.0

1. Q and P are subalternates, since Q implies P, this can be written as $Q \Longrightarrow P$ or $Q \supset P$.
2. P and Q are contradictory, since exactly one must be true.
3. P and Q are contraries, since if one of the pair is true, the other must be false, but both could be false if the week day were Wednesday.
4. P and Q are subcontraries, since if one of the pair were false, then the other must be true, however both could be true (for example, if Jane were 25).
5. P and Q are subalternates, since P implies Q, this can be written as $P \Longrightarrow Q$ or $P \supset Q$.

Answers Exercises 4.0

1. Some students did not pass the class.
2. All days were accounted for by the journal entries.
3. At least one person (or some people) were found to hold an illegal passport.
4. No cars were recalled by the manufacturer.
5. Yes they are, by noting that "some" has been replaced by "sometimes" and realizing that "No winters are colder than normal" is equivalent to, "Winters are never colder than normal".
6. No they are not, since it is possible for both to be false if the class were to contain 35 students and just *one* did not pass.

Answers Exercises 5.0

1. (Using the hint). Let $S = \{1, 3, 7, 9, 11\}$ and $P = \{\text{even numbers less than } 20\}$, with the Universal Set $U = \{\text{whole numbers greater than zero but less than } 20\}$.

We want to give an informal argument that "No S are P" implies that "Some S are not P", expressed symbolically we want to that $S \cap P = \emptyset$ implies $S \cap P^c \neq \emptyset$.

Clearly $S \cap P = \emptyset$ since no element of S is an even number. Also observe that the $P^c = \{1,3,5,7,9,11,13,15,17,19\}$, so $S \cap P^c = \{1,3,7,9,11\} = S \neq \emptyset$.

2. The picture on the left is a visual representation of a case where "No S are P" is true and in particular where S is non-empty (since we have assumed the Aristotelean view), whereas the statement on the right represents $S \cap P^c$ (the shaded area is not part of the elements of $S \cap P^c$). Since S is not empty, and contained in $S \cap P^c$, then we have $S \cap P^c \neq \emptyset$. In other words, "No S are P" implies the truth of "Some S are not P" when the set S is non-empty.

3. To argue informally in the way we have been doing, we need to choose two typical sets S and P, where S is not empty, and show that "Some S are P" and "Some S are not P" are subcontraries. To do this we must show that these statements satisfy the following two conditions:

a) both can not be false
b) it is possible for both to be true.

Let our Universal set be as before, and let $S = \{1, 3, 7, 9, 11\}$ and $P = \{2, 3,4,6, 7, 10\}$.

Recall that as sets, "Some S are P" is equivalent to $S \cap P \neq \emptyset$, and "Some S are not P" is equivalent to $S \cap P^c \neq \emptyset$.

Let us first show that condition b) is satisfied.

This is easy, since $S \cap P = \{3,7\} \neq \emptyset$, and $S \cap P^c = S \neq \emptyset$, so both statements can clearly be true, and it follows immediately that both can not be false under this choice (since both are actually true under our choice of sets).

4. First observe this is formally proven in section 6.0. Here we just show that it can be true for a choice of two normal sets (which is not a formal proof, since formal proofs have to show these statements are true for any choice of sets, hence we can't just show it is true for only one choice).

Using the hint, let $S = \{2, 4, 6, 8, 10\}$, $P = \{2,4,6,8,10,12,14,16,18\}$ and $U = \{\text{whole numbers greater than zero but less than 20}\}$. Since P^c is all odd numbers greater than 0 but less than 20, we have $S \cap P^c = \emptyset$. But since S is a subset of P, we have $S \cap P = S$.

Answers Exercises 6.0

1. **Theorem:** The I and O propositions are not necessarily subcontraries under the Boolean definition.

We want to show that at least one criterion for two propositions to be subcontraries given by the definition does not hold. In this case, we will show that both $S \cap P \neq \emptyset$ and $S \cap P^c \neq \emptyset$ can both be false. Since our theorem deals only with the Boolean interpretation, the first line in the proof re-iterates this.

Proof: Suppose we are using the Boolean definition and that the subject class S is empty. Consider the statement $S \cap P$, we want to show $S \cap P = \emptyset$. Since $S = \emptyset$ we have,

$$
\begin{aligned}
S \cap P &= \emptyset \cap P && \text{since we assumed } S = \emptyset \\
&= P \cap \emptyset && \text{by the commutative laws} \\
&= \emptyset && \text{by theorem 8}
\end{aligned}
$$

Hence this shows that $S \cap P \neq \emptyset$ is false, because we have proven that $S \cap P = \emptyset$. A similar argument shows that $S \cap P^c = \emptyset$, showing that both $S \cap P \neq \emptyset$ and $S \cap P^c \neq \emptyset$ can be false.

2.

Proof: Suppose we are using the Aristotelian definition so that the subject class S is not empty but the predicate class P is empty. We want to show it is not possible for $S \cap P \neq \emptyset$ and $S \cap P^c \neq \emptyset$ to be false. First, to show $S \cap P \neq \emptyset$ is false we need to show that $S \cap P = \emptyset$, but this follows immediately under our assumption that P is empty and theorem 8. Observe that since $P = \emptyset$, by theorem 3 we have, $P^c = U$ where U is the Universal set. Since we are using the Aristotelian definition S can not be empty, hence it is possible to choose sets S and P such that $S \cap P^c \neq \emptyset$ (just let S be a subset of P^c for instance).

3. Since we are given that $S \cap P = \emptyset$ is *true* for ***any choice*** of set S, then we can choose S to be U, the universal set. Since P must be a subset of the universal set (by definition of the universal set) and $U \cap P = \emptyset$ means P must be the empty set. Since P is empty, then no element of the set S (for any choice of S) can be an element of P.