Sets

Basic Definitions, Operations, Theorems and Examples

Sets

Definition: A **set** is a well-defined collection of objects with a common property or properties. The individual objects are called *elements* or *members*.

When we stipulate that a set is well defined, this just means there must be some way of specifying the properties that members of the set share which allows for an unambiguous interpretation. Also there is nothing in the definition of a set that prevents a set from containing no objects at all. When this occurs, we say the set is the **empty set** or the **null** set, which we will denote as $\emptyset$, although some texts use { } to denote the empty set. Note that $\{\emptyset\}$ is not the empty set, but rather the set which *contains* the empty set as an *element*.

An example of a set would be the collection of all numbers which are even but greater than zero and less than 10. We could write this set as:

{2,4,6,8}

Many times it is convenient to shorten the names of sets by referring to them with a letter. This allows us to write:

$S = \{2,4,6,8\}$

When we speak of the set S later on, it will be understood to mean {2, 4, 6, 8} until we designate S to mean another collection of objects.

Notice that the following two sets are the same {6,2,8,4} and {2,4,6,8}, since sets contain elements not elements ordered in a specific way. Different orderings of the elements in a set do not change the nature of the set. One reason this is the case is that sometimes there is no obvious way to order elements in a set - think of the set of all red objects in one room in your house – is there any obvious and universal way to order those objects? For this reason, when we want to speak of ordered objects (for objects that have an obvious ordering), we use something called a **list** rather than a set. We will not worry about ordered lists of elements in this introduction.

We should also note that the elements of some sets are other sets, like the set of sets of schools in different counties in a given state. Since sets themselves can be the members of a given set, this leads to the question as to whether a set can be an element/member of itself? This odd question has posed several problems to logicians and mathematicians alike. So much so, that many definitions of the term "set" forbid a set to be an element of itself since allowing it to be can lead to **paradoxes** (sentences which seem to allow one to conclude they are false if assumed to be true, or true if assumed to be false). For our purposes in this introductory treatment we will never have to worry about such complications. [1]

When listing the individual elements of a set each element is separated from the next by a comma. The elements are enclosed between two curly braces – duplicate objects are omitted.

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[1] The book, Introduction to Graph Theory by R. Trudeau has a very accessible and good treatment of the subject. Also note that allowing sets to be members of themselves need not always pose a problem. Think of "the set of all things which can be described in 12 words".

Sets assume a **universal set**, which is a set which contains **all possible elements** you might wish to consider. The universal set will always be denoted by the letter U. Sometimes this universal set is obvious and omitted. Otherwise the universal set U is specified. Many times more than one choice can be made for the universal set.

Examples:

S = {1, 2, %, 7,*, 3, a, Y, -,?, Z}	Here the universal set could be the set of all typographical signs
A = {mammals}	Here the universal set might be the set of all living things
B = {things which are both red and colorless}	Since nothing has this property, B is just the empty set, so we could choose any set to be the universal set

Set Notation

A common alternative to listing elements of a set is to define a set by using the following **set notation**, where the symbol "∈" means "is a member/element of", or just "in", and the symbol "|" is short hand for "such that" (sometimes a colon ":" in used instead of "|"). Here is an example:

$$S = \{x \in B \mid x \text{ has a list of specific properties}\}$$

This is read in English as, "S is *in* the set B *such that* these elements have the following list of specific properties". Observe that x in the above notation is just a short hand symbol that stands for an element in the set B with the stated properties, and our set S is the totality of all such elements x.

In general we have the following:

$$\underbrace{S}_{\text{name of our set}} = \left\{ \underbrace{x \in B} \;\middle|\; \underbrace{x \text{ has a list of specific properties}} \right\}$$

- $x \in B$: states where the elements of our set are taken. B is always a subset of U our Universal Set. The symbol ∈ means "in" or "is an element of".
- x has a list of specific properties: denotes a list of specific properties that each element x of our defined set S must have. Normally these are a list of properties contained in some subset of elements of B. But we allow S to be empty, hence it is permissible to list a property that no element of B has, in which case the set S is empty.

Recall the symbolic way to say "x is an element of S" is to write, $x \in S$.

Examples:

$S = \{x \in \text{Set of whole numbers} > 0 \mid x \text{ is even}\}$

The set S contains all elements *x* in the set of whole numbers greater than zero *such that x* is even.

$A = \{x \in \text{living things} \mid x \text{ has mammary glands}\}$

The set A contains all members of living things *such that* they have mammary glands.

$B = \{x \in \text{set of numbers} \mid x > 0 \text{ and } x < 0\}$

The set B contains all elements in the set of numbers (*such that they*) are greater than zero and less than zero. Can you find any numbers which have this property?

Subsets

Definition: A set S is a **subset** of another set P if and only if every element of S is also an element of P. We denote that S is a subset of P by writing $S \subseteq P$ and read this as, "S is an (im)proper subset of P"

Observe that the definition allows P to be equal to S , since such condition obviously satisfies the definition of being a subset. In such cases we say the set P is an **improper** subset of P This is perhaps an unfortunate choice of terms, but it has been in use for years, and we are stuck with it now. Notice that if S is a subset of P, then every element of S is also P. Many times whenever S is a subset of another set P, we say S is **included** in P. Also by definition, every set is a subset of the universal set U.

Examples:

a) Let S = {1,2,4,6}, and let P = {set of all whole numbers}, then $S \subseteq P$ (read "S is a subset of P").

b) Let S = {dogs} and let P = {mammals}, then $S \subseteq P$.

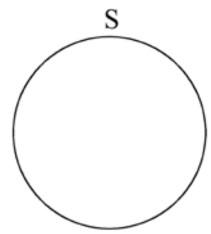
c) Let S = {1,2,4,6,8,10} and let P = {set of even numbers), in this case, S is **not** a subset of P, since every element in S is not also an element of P.

d) Let S = {1} and let P = {1}, then $S \subseteq P$. (In this case S is a subset of P)

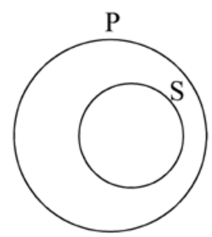
e) Let S = {1, 3, 7, 23} and let S = {1, 3, 7, 25} then S is not a subset of P, since not every element in S is also an element is P.

Visual Representations of Sets and Subsets

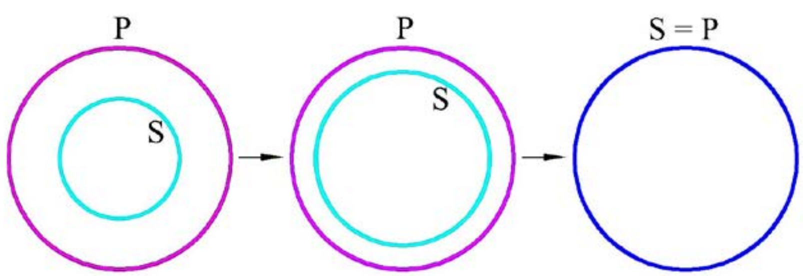
By using circles as a graphic representation of a set where one imagines what is "contained" within the circle to be the elements of the set, we can represent a set S this way:



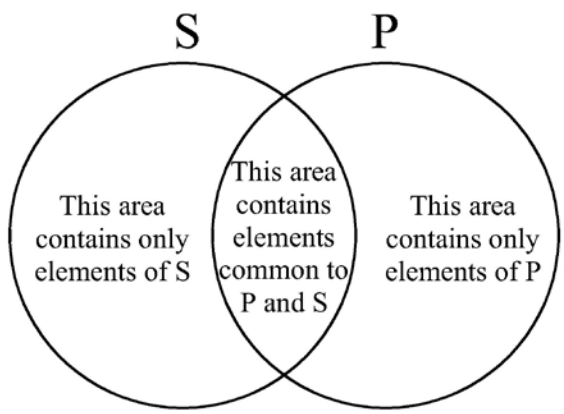
With this in mind a natural way to represent that S is a subset of P is to depict a smaller circle S contained wholly within a larger circle P. This "shows" that every element of S is also an element of P, which means (by definition) that S is a subset of P.



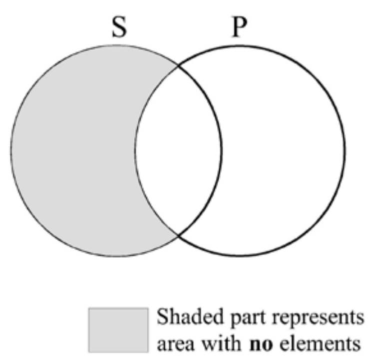
By increasing the size of the inner circle to equal the size of the outer circle (and making sure their centers coincide) we can illustrate that S is a subset of itself denoted by $S \subseteq S$. By coloring the boundaries of our sets, and imagining S to increase in size to become equal to P, we can visualize when it is the case that S is a subset of itself.



Many times we will use two overlapping circles to visually represent relationships between two sets. When two circles overlap we have the following three areas:



By shading in areas of the circle which have **no** elements, we can represent that S is a subset of P this way:



Notice that the area that contains only elements of S has been shaded in, indicating that there are no elements of S in this area, which means that all the elements of S are contained in the area where elements common to both S and P are found. Hence, visually every element of S is also in P, which means that S is a subset of P.

Important Example: Why the empty set is a subset of any set.

The fact to the empty set turns out to be a subset of *any* set often confuses students. The reason stems from the conditions that make "if… then" statements true. Take a conditional statement, "If P, then Q" which we can express symbolically as "$P \supset Q$", and suppose P (the antecedent) is false. We then have two cases determined by the value of Q, either :

$F \supset T$ or $F \supset F$

Looking at the truth table for conditional statements, we find that $F \supset T = T$ and $F \supset F = T$. Hence we can conclude that whenever the antecedent of a conditional statement is FALSE that is a sufficient condition to conclude the **entire conditional statement is true.**

Now let's apply this to the case of subsets. We say S is a subset of P if the following condition is true:

If x is an element in S **then** x is also an element in P.

Notice this is a *conditional statement* that gives the requirement for S to be a subset of P.

Now suppose that S is the empty set, then the antecedent, "x is an element in S" is ALWAYS false because S is empty, x is never an element of S.

From what we observed above, whenever the antecedent of a conditional statement is false, then the conditional statement as a whole is true.

This means that when S is empty it meets the requirement to be a subset for ANY set P, hence the empty set is a subset of every set!

Mentally we can picture the fact that an empty set is a subset of any set P by imagining a circle S inside a larger circle P, as S shrinks to a point (which contains nothing), S is still inside P, hence S is a subset of P. This visual aid is not a rigorous proof but it may help conceptualize this unexpected property of the empty set.

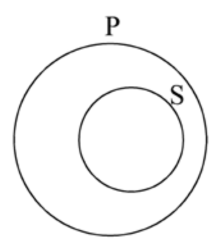
Proper Subset

Definition: A set S is a **proper subset** of another set P if and only if every element of S is also an element of P *and* S is not the same set as P. We denote that S is a proper subset of P by writing $S \subset P$.

Sometimes it is important to specifically state S is only a proper subset of P. For our purposes we will seldom, if ever, need this extra concept. Observe that in the examples above all are proper subsets except example *d*.

Visual Representation that S is a Proper Subset of P

This can be represented in the following way, keeping in mind that the set S never enlarges to the size of P in this case.



Let's continue with our definitions by giving some important operations on Sets.

Set Difference

Definition: Let P and S be two sets. Then then the **set difference** between P and S, denoted as P – S (or sometimes P\S) is the set which contains everything in P which is not in S.

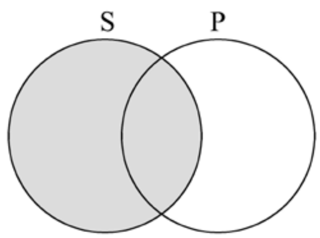
Examples:
a) Let P = {whole numbers}, let S = {even numbers}. Then P – S = {odd numbers}

b) Let P = {1,2,5,7,9,10} and let S = {5,9}, then P – S = {1, 2,7, 10}, and $S - P = \emptyset$, since S is a subset of P.

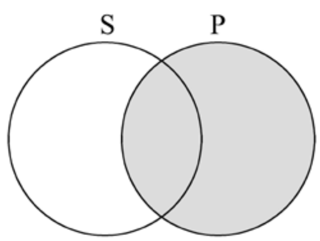
c) Let P = {1,2,5,7,9,10} and let S = {3, $, #, π, "dog"}, then P – S = P, because S contains no elements in common with P.

d) Let P = {1,2,5,7,9,10} and let S = ∅, then P – S = P (for the same reason as above).

We can illustrate $P - S$ by using our overlap of P and S and shading all of S, indicating the only remaining elements to be considered are those in P but not in S.



Using this same technique, we get $S - P$ as:



The above examples illustrate an interesting fact about set difference. In general $P - S \neq S - P$, since there is no reason to suppose what is contained in the unshaded areas is the same in both cases.

The Compliment of a Set

Definition: Let U be the specified Universal set, and let P be any subset of U, then the **compliment of P** is U – P (all the elements in U not contained in P). The compliment of P is denoted by P^C or many times $\bar{P}$, we will use the former symbol as it is easier to reproduce in a word processor. Although you should be familiar with both symbols.

Notice that we need the Universal set to be specified to avoid multiple answers. For example, if our set P = {mammals}, then U could be the set of all living things, or the set of all animals, or the set of things in the solar system. In each case P^C would be different. Hence, if you are asked to find the compliment of a set, you will need to be given the Universal set U where P "resides". To make this more clear, P^C is sometimes called the "compliment of P with respect to U".

Examples:
a) Let U = {whole numbers greater than zero but less than 20}

b) Let P = {1,2,3,4,5,6,7,8,9,12}, then $P^C = U - P = \{10,11,13,14,15,16,17,18,19\}$

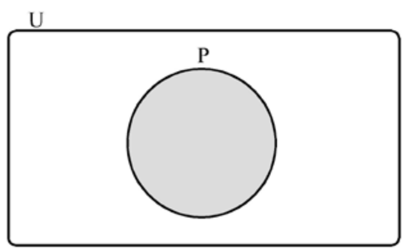
c) Let U = {1,3,%,&,*, *jkz*}, let P = {3}, then $P^C = U - P = \{1, \%, \&, jkz\}$

Notice that if our Universal set is infinite in size, such as the set of all whole numbers, then it can be the case that P^C is also infinite in size too. Hence, to write out P^C as a list of elements would be impossible, as the list would be infinite. Sometimes it is possible to write infinite sets which have an obvious pattern to their elements by using an ellipsis (a set of three dots), such as S = {1, 2, 3, …} In this case, S is an infinite set containing all the positive whole numbers (the pattern is supposed to be self-evident).

The normal way to get around this problem is just to write things out in Set Notation, where the defining elements of our set is given.

Visual Representation of Set Compliment

Since every set is a subset of the Universal set, the customary way to visualize the compliment of any set is to "embed" the set in the universal set U and shade out everything in P.



Set Union

Definition. Let S and P be any two sets. The **union** of S and P, denoted S ∪ P is the set that contains all the elements in S and all the elements in P.

In taking unions of sets, it is tempting to list elements twice (since we are just throwing together everything in both sets) – this is bad notation and practice, so be careful not to do it (it is bad, since we often need to know the size of a set, and if a set has duplicate elements, then we can't easily determine the set size by just counting).

Examples:

a) Let P = {whole numbers}, let S = {even numbers}

Then $P \cup S = \{\text{whole numbers}\}$ (this is true since S is a subset of P)

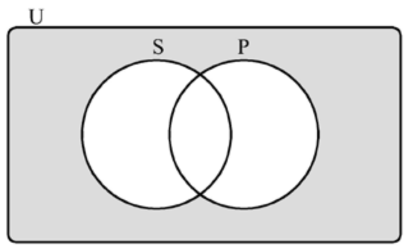
b) Let P = {1,2,5,7,9,10} and let S = {5,109}, then $P \cup S = \{1,2,5,7,9,10, 109\}$

c) Let P = {1,2,5,7,9,10} and let S = {3, $, #, π, "dog"}, then $P \cup S$= {1,2,5,7,9,10, 3, $, #, π, "dog"}

d) Let P = {1,2,5,7,9,10} and let $S = \emptyset$, then $P \cup S = P$ (for the same reason as above).

Visual Representation of Set Union

Again we turn to our overlapping circles to represent set Union, embedding both sets S and P in the universal set U. The union is then everything contained in both S and P, but not elements in U not in S or P, which can be represented by shading the area outside of S and P.



Set Intersection and Disjoint Sets

Definition: Let S and P be two sets. The **intersection** of S and P, denoted $S \cap P$ and read "S intersect P" is the set that contains all elements common to both S and P. Stated another way, if x is an element of $S \cap P$, then x is an element of S *and* x is an element of P.

Observe that S and P can be any two sets, including the Universal set U or the empty set $\emptyset$.

Examples:

a) Let P = {whole numbers}, let S = {even numbers}. Then $P \cap S$ = {even numbers} (this is true since S is a subset of P).

b) Let P = {1,2,5,7,9,10} and let S = {5,109}, then $P \cap S$ = {5}

c) Let P = {1,2,5,7,9,10} and let S = {3, $, #, π, "dog"}, then $P \cap S = \emptyset$.

d) Let P = {1,2,5,7,9,10} and let $S = \emptyset$, then $P \cap S = \emptyset$.

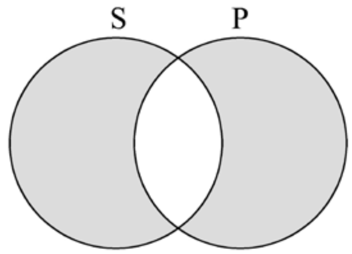
Disjoint Sets

Definition: If the **intersection** of two sets S and P *is empty,* then we say S and P are **disjoint.**

The sets P and S in examples *c* and *d* above are both disjoint. Another way to express this definition is to note that two sets are disjoint if they have no elements in common meaning no member of S is an element of P and no member of P is an element of S.

Visual Representation of Set Intersection

Overlapping circles work well whenever the set intersection in not empty. When the intersection is empty, two non-overlapping circles work. Since the latter is easily imagined, we just show the former.



Set Size

Definition: We denote the *total number* of elements in a set S by the following notation $\#S$, and say #S is the **size of the set** S. (Many books use $|S|$ for this notation rather than #S)

Observe $\#S$ is zero if S is the empty set, otherwise $\#S$ is a positive whole number if the size of S is finite. Some sets are infinite in size (such as the set of even numbers), in which case we say the size of the set is infinite.

Here notation can be confusing. If S and P are two finite sets, then the notation $\#S + \#P$ makes sense. The result is a positive whole number or zero. In other words if S is a set, then #S is a whole number, zero or infinity. Since addition using infinity is not defined, the statement $\#S + \#P$ may not make sense. Be careful when writing about the size of two sets, and do not write the nonsensical S + P when you really mean #S + #P!

Examples:

a) Let P = {whole numbers}, then #P is infinite.

b) Let P = {1,2,5,7,9,10} then #P is 6, since P contains 6 elements.

c) Let P = {1,2,5,7,9,10} and let $S = \emptyset$, then $P \cap S = \emptyset$ and $\#(P \cap S) = 0$

Basic Proofs and Other Theorems about Sets

We will list several theorems about sets. We take the time to prove one of these relationships and leave the rest without proof as general proofs of all of these theorems are abundant in the literature on sets. We will use them to prove other non-obvious facts about sets. An interesting and fun exercise is to try to prove some of these on your own, either informally with pictures, or formally using definitions and following the example proofs we give below.

Some basic Theorems about Sets

1. $(P^C)^C = P$	The compliment of a set compliment is the original set.
2. $U^C = \emptyset$	The compliment of the Universal set is the empty set.
3. $\emptyset^C = U$	The compliment of the empty set is the Universal set.
4. $P - \emptyset = P$	The empty set subtracted from any set is just that set.
5. $P - P = \emptyset$	Any set subtracted from itself is the empty set.

Some basic Theorems about Sets

6. $P - S = P \cap S^C$	S subtracted from P is the intersection of P with the compliment of S.
7. $P \cap U = P$	The intersection of P and the universal set is P.
8. $P \cap \emptyset = \emptyset$	The intersection of P and the empty set is the empty set.
9. $P \cap P^c = \emptyset$	The intersection of P and the compliment of P is the empty set.
10. $P \cap P = P$	The intersection of P with itself is P
11. $P \cup U = U$	The union of P and the universal set is the Universal set.
12. $P \cup \emptyset = P$	The union of P and the empty set is P.
13. $P \cup P^c = U$	The union of P and the compliment of P is the universal set U.
14. $P \cup P = P$	The union of P with itself is P.

Commutative Laws

15. $P \cup S = S \cup P$	The operation of taking unions commutes.
16. $P \cap S = S \cap P$	The operation of taking intersections commutes.

Associative Laws

17. $P \cup (S \cup R) = (P \cup S) \cup R$	The operation of taking unions is associative.
18. $P \cap (S \cap R) = (P \cap S) \cap R$	The operation of taking intersections is associative.

Distributive Laws

19. $P \cup (S \cap R) = (P \cup S) \cap (P \cup R)$	The set P distributes across the set $S \cap R$
20. $P \cap (S \cup R) = (P \cap S) \cup (P \cap R)$	The set P distributes across the set $S \cup R$

DeMorgan's Laws for Sets

21. $(P \cup S)^C = P^C \cap S^C$

22. $(P \cap S)^C = P^C \cup S^C$

23. $P - (S \cup R) = (P - S) \cup (P - R)$

24. $P - (S \cap R) = (P - S) \cup (P - R)$

In the above list, P , S and R designate any three sets, not necessarily distinct (which means the theorems hold true even if we treat S, P and R as the same set). However U is always the universal set. Hence since $(R \cup M)$ is a set and $(A \cap B)$ is a set, commutation tells us that $(R \cup M) \cup (A \cap B) =$

$(A \cap B) \cup (R \cup M)$ and so forth. In general, any single letter that designates a set in the above list can be replaced by any string of set operations, as long as the same string of set operations is used to replace the same letter.

Since the nature of logic is to provide tools to prove arguments (which we will now call theorems), we need to pause for a moment and pose the question: How does one prove two sets are equal? Before we answer that question, we need to be clear on what it means for two sets to be equal. We will then state this equality in terms of our set operations. Set equality is the easy part – as its definition is exactly as one might expect.

Definition: Two **sets are equal** if they have the same elements.

Suppose P and Q are two equal sets , the every element of P is an element of Q, and every element of Q is an element of P. That means that both of the following statements are true:

$$P \subset Q$$

$$Q \subset P$$

In words, P is included in Q, and Q is included in P. This "double" inclusion is the standard way we prove two sets are equal. We start by considering an arbitrary element in P, and show it is also in Q. Then we do it again, with the sets switched and start by considering every element in Q and show it is also in P. Here I should mention, that when we, "start by considering an arbitrary element in" we do not mean that we look at each element individually (unless that is literally the only way), but rather look at the defining properties that tell us something about every element of the set . Using this method, let's prove Theorem 1 from our list.

Theorem 1: $(P^C)^C = P$

Recall we want to show that $(P^C)^C \subset P$ and that $P \subset (P^C)^C$. Our proof will start by considering any arbitrary element x in $(P^C)^C$ and show that x is also an element of P. We will then repeat the same procedure going the other way around by considering any arbitrary element x in P and showing that x is also an element of $(P^C)^C$. Our main goal will be to use the definitions of set compliment.

Proof: First we show that $(P^C)^C \subset P$. Let x be any element in $(P^C)^C$. By definition P^C is the set of all elements not in P, so the complement of $(P^C)^C$ is the set of all elements which are not "not in P", but the elements which are not in P are just those elements in U not contained in P, so the elements not "not in P" are just the elements contained in P, so x is in P, which is what we wanted to show.

Now we need to show that $P \subset (P^C)^C$. Let x be any element of P, then x is also in $(P^C)^C$ by the same argument as given above. Since $(P^C)^C \subset P$ and $P \subset (P^C)^C$, we have $(P^C)^C = P$ which is what we wanted to show.

Now let's use these Theorems directly to prove some properties of sets.

Theorem 25: If $A = B$, then $A \cap B = A \cup B$

Proof: By theorems 10 and 14 we have $A = A \cap A$ and $A = A \cup A$, but since we have supposed that A = B, we have $A \cap B = A \cup B$ which is what we wanted to show.

Notice that the key to the proof was the assumption that $A = B$ and theorems 10 and 14. Also notice that theorems 1 - 24 are statements about set equality, hence we did not have to use the method of double inclusion to prove that $A \cap B = A \cup B$ since the equalities in theorems 10 and 14 did the work for us.

Theorem 26: $(S \cap U) - (S \cap P) = S \cup (U - P)$

We start our proof by transforming the left hand side of the equality step by step using Theorems 1 – 24 until our last step is the right hand side of the equation.

Proof:

$(S \cap U) - (S \cap P) =$

$= (S \cap U) \cap (S \cap P)^C$	by Theorem 6
$= (S \cap U) \cap (S^c \cup P^c)$	by Theorem 22
$= ((S \cap U) \cap S^c) \cup ((S \cap U) \cup P^c)$	by Theorem 19
$= (S \cap S^c) \cup (S \cup P^c)$	by Theorem 7
$= \emptyset \cup (S \cup P^c)$	by Theorem 9
$= (S \cup P^c)$	by Theorem 12
$= S \cup (U - P)$	by definition of P^c

Hence $(S \cap U) - (S \cap P) = S \cup (U - P)$ which is what we wanted to show.

You should verify the use of each theorem in the proof above with the possible exception of the use of Theorem 19, and make sure you understand how it was used.

At this point some students are curious as to how one knows which of the 24 theorems to use. It would be nice if there were some easy way to decide where to start and which theorem to use, but the actual answer is that proving things takes practice. Trial and error are an inevitable part of the process, but practice helps a lot. Speaking of which, it is now time to try the practice exercises!

Exercises:

Let P = {1,2,6,21,135}, let S = {20,21,22}, let M = {2,21}, and let A = {19,29,31}.

a) What is $P \cap S$?

b) What is $P \cup S$?

c) What is $P - S$?

d) What is $S - P$?

e) If $U = \{x \in \text{Whole numbers less than } 200 \mid x \text{ is between } 19 \text{ and } 30\}$, then what is S^c?

f) What is $\#P - \#S$?

g) What is $\#S - \#P$?

h) What is $M \cap A$?

i) What is $\#(M \cap A)$?

j) What is $(M \cap A)^c$

Answers:

a) {21}
b) {1,2,6,21,135,20, 22}
c) {1,2,6,135}
d) {20,22}
e) {23,24,25,26,27,28,29}
f) 5 – 3 = 2
g) 3 – 5 = -2
h) ∅
i) 0
j) The universal set U or $M^c \cup A^c$

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