19.1 Problem 5.

Define the function $f : \mathbf{I} \to \mathbb{R}$, where $\mathbf{I} = [0, 1] \times [0, 1]$ by

$$f(x,y) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 2y & \text{if } x \text{ is irrational} \end{cases}$$

Show that f is not integrable.

Recall **Fubini's Theorem**, which states: Suppose that the function $f : \mathbf{I} \to \mathbb{R}$ is integrable, where $\mathbf{I} = [a, b] \times [c, d]$ in $\mathbb{R}^2$. For each point x in $[a, b]$, define the function $F_x : [c, d] \to \mathbb{R}$ by $F_x(y) = f(x, y)$ for $y \in [c, d]$, suppose that the function $F_x : [c, d] \to \mathbb{R}$ is integrable, and define

$$A(x) = \int_c^d f(x,y)dy$$

Then the function $A : [a, b] \to \mathbb{R}$ is integrable, and

$$\int_{\mathbf{I}} f = \int_a^b A(x)dx = \int_a^b \left[\int_c^d f(x,y)dy \right] dx$$

Our proof will use the contrapositive of this theorem and the fact that Fubini's theorem being perfectly general requires that one obtain the same result when the order of integration is reversed.

Proof. First we evaluate $\int_a^b \left[\int_c^d f(x,y)dy \right] dx$. Observe that for each point $x_0 \in [0, 1]$ and $y \in [0, 1]$ we have $F_{x_0}(y) = 1$ or $F_{x_0}(y) = 2y$ according to whether x_0 is rational or irrational, in either case both are continuous hence integrable, so we can apply Fubini's theorem to obtain $\int_0^1 f(x,y)dy = 1$. Hence $A(x) = 1$, so $\int_{\mathbf{I}} f = \int_0^1 A(x)dx = \int_0^1 dx = 1$.

Now evaluating $\int_0^1 \left[\int_0^1 f(x,y)dx \right] dy$ note that for each fixed $y_0 \in [0, 1]$, and for $x \in [0, 1]$ we have $F_{y_0}(x) = 1$ if x is rational or $F_{y_0}(x) = 2y_0$ if x is irrational. Since the values of these two functions only agree when $y_0 = \frac{1}{2}$, the inner integral $\int_0^1 f(x,y)dx$ can not possibly exist. Since we obtain different results by changing the order of integration, by the contrapositive of Fubini's theorem we conclude that $f : \mathbf{I} \to \mathbb{R}$ is not integrable. □

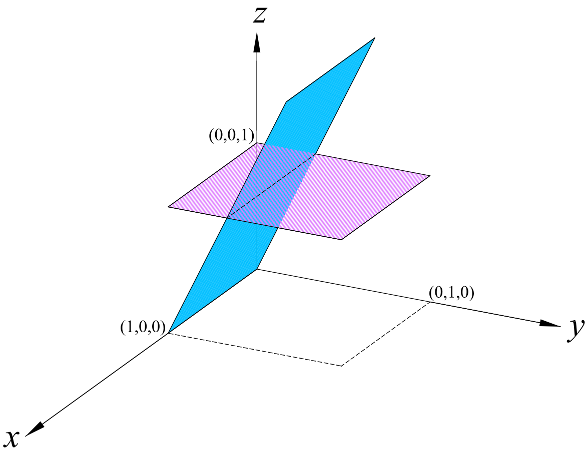


Figure : $f(x,y) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 2y & \text{otherwise} \end{cases}$

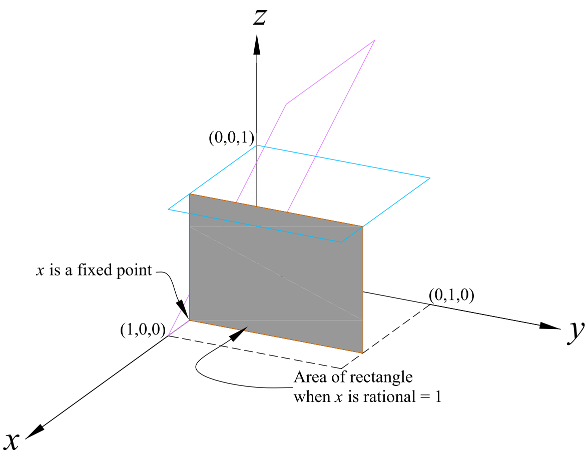


Figure : $A(x) = \int_0^1 f(x,y)dy = \int_0^1 1dy = 1$ when x is rational

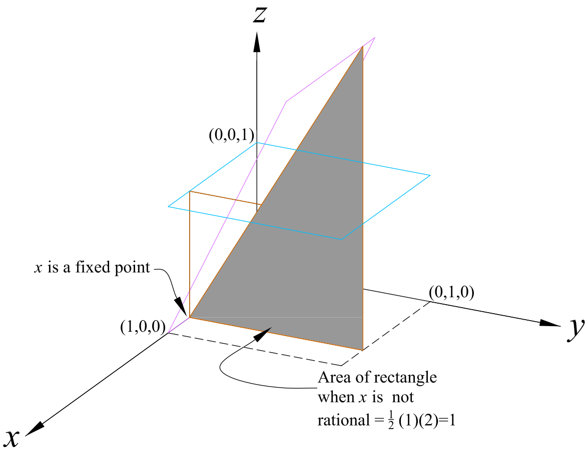
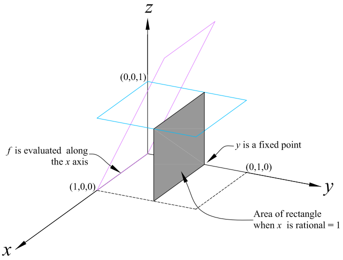


Figure : $A(x) = \int_0^1 f(x,y)dy = \int_0^1 2ydy = 1$ when x is not rational



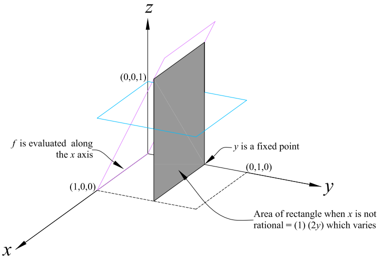


Figure : $B(y) = \int_0^1 f(x,y)dx$. Notice x takes on both rational and irrational values, hence $f(x,y) = 1$ and $2y$ the upper and lower Darboux sums don't generally agree under any partition $P = (P_1, P_2)$