Preliminaries.

We will start with what it means for a subset of $\mathbb{R}^n$ to be *separated.* To help, we will just look at a subset A of $\mathbb{R}^2$, since that is easiest to visualize.

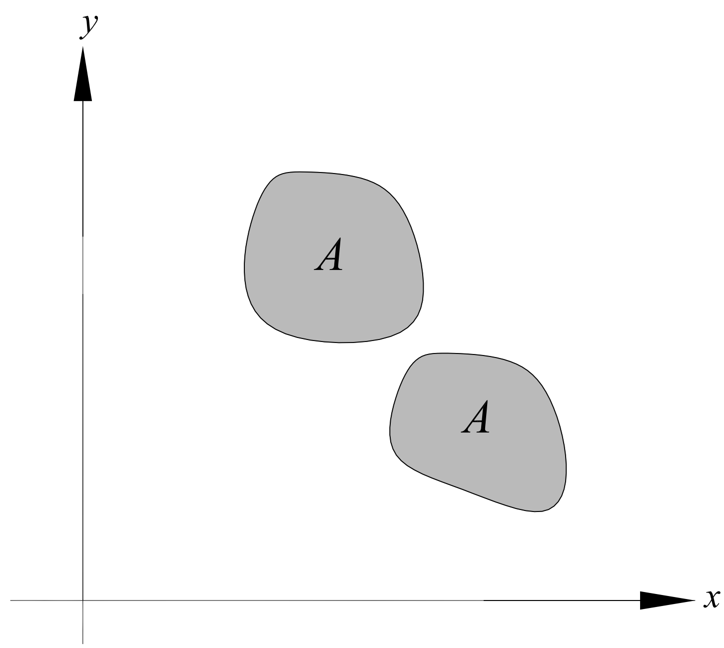
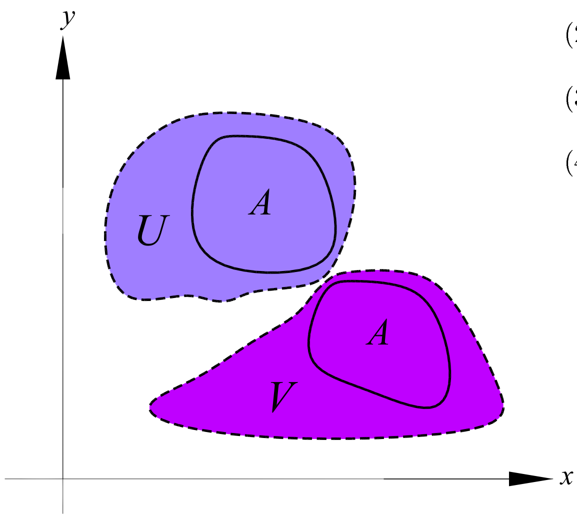


FIGURE 1. A subset of $\mathbb{R}^2$

Two *open* subsets $\mathcal{U}$ and $\mathcal{V}$ of $\mathbb{R}^n$ are said to **separate** A provided that the following hold:

(1) $\mathcal{U} \cap A \neq \emptyset$, the Intersection of $\mathcal{U}$ with A is non-empty.
(2) $\mathcal{V} \cap A \neq \emptyset$, the Intersection of $\mathcal{V}$ with A is non-empty.
(3) $(\mathcal{U} \cap A) \cap (\mathcal{V} \cap A) = \emptyset$, $(\mathcal{V} \cap \mathcal{A})$ and $(\mathcal{U} \cap \mathcal{A})$ are disjoint
(4) $(\mathcal{U} \cap A) \cup (\mathcal{V} \cap A) = A$, the union of $(\mathcal{V} \cap \mathcal{A})$ with $(\mathcal{U} \cap \mathcal{A})$ make up A

So let's separate A into two subsets that satisfy the above requirements:



(1) $\mathcal{U} \cap A \neq \emptyset$, the Intersection of $\mathcal{U}$ with A is non-empty.
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(4) $(\mathcal{U} \cap A) \cup (\mathcal{V} \cap A) = A$, the union of $(\mathcal{V} \cap \mathcal{A})$ with $(\mathcal{U} \cap \mathcal{A})$ make up A

FIGURE 2. $\mathcal{U}$ and $\mathcal{V}$ separate A

In contrast, a subset A of $\mathbb{R}^n$ is said to to be **connected** provided that there do not exist two open subsets of $\mathbb{R}^n$ that separate A.

Section 11.4, problem 4.
Suppose that A is a subset of $\mathbb{R}^n$ that fails to be connected, and let $\mathcal{U}$ and $\mathcal{V}$ be open subsets of $\mathbb{R}^n$ that separate A. Suppose that B is a subset of A that is connected. Prove either $B \subseteq \mathcal{U}$ or $B \subseteq \mathcal{V}$.

Proof. We will argue by contradiction, assuming that it's not the case that $B \subseteq \mathcal{U}$ or $B \subseteq V$, but B is connected and derive a contradiction.

Suppose that it is not the case that $B \subseteq \mathcal{U}$ or $B \subseteq \mathcal{V}$, then there exists $b_1 \in B$ which is not in $\mathcal{U}$ but since $B \subseteq A$, we have $b_1 \notin (\mathcal{U} \cap A)$ so $b_1 \in (\mathcal{V} \cap A)$. Similarly we have $b_2 \in B$ which is not in $\mathcal{V}$, so $b_2 \in (\mathcal{U} \cap A)$, hence $\mathcal{U} \cap B \neq \emptyset$ and $\mathcal{V} \cap B \neq \emptyset$.

Again, using the fact that $B \subseteq A$, then $(\mathcal{U} \cap A) \cap (\mathcal{V} \cap A) = \emptyset \Rightarrow (\mathcal{U} \cap B) \cap (\mathcal{V} \cap B) = \emptyset$, and $(\mathcal{U} \cap A) \cup (\mathcal{V} \cap A) = A \Rightarrow (\mathcal{U} \cap B) \cup (\mathcal{V} \cap B) = B$, hence B fails to be connected, contrary to supposition. So either $B \subseteq \mathcal{U}$ or $B \subseteq \mathcal{V}$.

□

We can prove problem 4 another way by using Theorem 11.36, which states that a subset of $\mathbb{R}^n$ is connected if and only if it has the intermediate value property. (Recall, a subset B of $\mathbb{R}^n$ has the *intermediate value property* provided that every continuous function $f : B \to \mathbb{R}$ has an interval as its image)

Proof. As before we assume to the contrary that it's not the case that $B \subseteq \mathcal{U}$ or $B \subseteq V$, but B is connected and derive a contradiction, this time by showing the existence of a continuous function $f : B \to \mathbb{R}$ whose interval is not an image.

Above we showed that our assumption that it is not the case that $B \subseteq \mathcal{U}$ or $B \subseteq V$, gave us $\mathcal{U} \cap B \neq \emptyset$ and $\mathcal{V} \cap B \neq \emptyset$, so for a point $\mathbf{w} \in B$, define f as follows:

$$f(\mathbf{w}) = \begin{cases} 0 & \text{if } \mathbf{w} \text{ is in } \mathcal{U} \cap B \\ 1 & \text{if } \mathbf{w} \text{ is in } \mathcal{V} \cap B \end{cases}$$

Then f is continuous, since both $\mathcal{U}$ and $\mathcal{V}$ are open subsets of $\mathbb{R}^n$, and for each point $\mathbf{w}$ in B, there is an open ball $\mathcal{B}_r(\mathbf{w})$ such that $f : B \to \mathbb{R}$ is constant on $B \cap \mathcal{B}_r(\mathbf{w})$, but the image of f is not an interval, but rather the set $\{0, 1\}$, hence by Theorem 11.36, B is not connected, contrary to supposition. $\square$